Retrieval phase diagrams of non-monotonic Hopfield networks

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Abstract. We investigate the retrieval phase diagrams of an asynchronous fully connected attractor network with non-monotonic transfer function by means of a mean-field approximation. We find for the noiseless zero-temperature case that this non-monotonic Hopfield network can store more patterns than a network with monotonic transfer function investigated by Amit *et al*. Properties of retrieval phase diagrams of non-monotonic networks agree with the results obtained by Nishimori and Opris who treated synchronous networks. We also investigate the optimal storage capacity of the non-monotonic Hopfield model with state-dependent synaptic couplings introduced by Zertuche *et al*. We show that the non-monotonic Hopfield model with state-dependent synapses stores more patterns than the conventional Hopfield model. Our formulation can be easily extended to a general transfer function.

1. Introduction

Statistical mechanical approaches were successful for the investigation of equilibrium properties of associative memories or attractor networks. The Hopfield model [1, 2] which updates its state asynchronously was investigated from a statistical mechanical point of view by Amit *et al* [3, 4], and a lot of interesting features were found. One of the main issues about the Hopfield model as an associative memory device is the critical storage capacity. Amit *et al* [3] showed that the Hebbian learning in the Hopfield model leads to the optimal storage capacity $\alpha_c = p/N = 0.138$, where p is the number of embedded patterns and N is the number of neurons. Fontanari and Köberle [5] extended the method of Amit *et al* to the synchronous networks and showed that the capacity remains the same $\alpha_c = 0.138$ and derived finite-temperature properties of synchronous networks. On the other hand, we cannot obtain information on the dynamical process of retrieval by equilibrium statistical mechanics.

Amari and Maginu [6] proposed a signal-to-noise ratio analysis to investigate the dynamical properties of synchronous networks. They divided the time-dependent local field $h_i^t = (1/N) \sum_j \sum_\mu \xi_i^\mu \xi_j^\mu \sigma_j(t)$ into a signal part m_t, which corresponds to the first ($\mu = 1$) term in the summation over μ, and a noise contribution N_i^t corresponding to the rest ($\mu \neq 1$). They assumed that the time-dependent noise term obeys the Gaussian distribution during the dynamical processes and showed that the capacity is $\alpha_c = 0.159$. Nishimori and Ozeki [7] pointed out by Monte-Carlo simulations that the assumption of a Gaussian distribution of the noise term is valid at least within statistical uncertainties if the final retrieval is successful. And they extended the Amari–Maginu theory to the network which updates its state stochastically and investigated the properties of the Hopfield network

at finite temperatures. The phase diagram obtained as the equilibrium limit of the extended Amari–Maginu dynamics is very similar to the phase diagram of Amit *et al* [3, 4].

The limitation of storing patterns in Hopfield networks comes mainly from the Hebbian interactions $J_{ij} = (1/N) \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$. In fact, Gardner [8–10] showed, in her pioneering papers, that the optimal storage capacity α_c is 2 for the general interaction J_{ij}. Many attempts have been made to increase the storage capacity of the Hopfield model to Gardner's limit $\alpha_c = 2$ by taking more complex synapses. Recently Zertuche *et al* [11] studied the storage capacity of the Hopfield model with state-dependent synapses by introducing a threshold parameter η. This parameter η determines which patterns contribute to the synapses. This synapse can be written as $J_{ij} = (1/N) \sum_\mu \xi_i^\mu \xi_j^\mu \Theta(m_\mu^2 - \eta^2/N)$. In their model, only patterns whose correlation with the state of the networks is greater or equal to the threshold are left finite to give a Hebbian contribution to the synapses. The capacity of the Hopfield network with this type of synapses is found to increase α_c from 0.138 to 0.171 at $T = 0$ and $\eta = 1.0$.

Nishimori and Opris [12] investigated the retrieval properties of an associative memory with a general transfer function using the extended Amari–Maginu theory [7]. They obtained the optimal storge capacity for the non-monotonic transfer function by taking the equilibrium limit of the recursion relation of the Amari–Maginu dynamics and showed that networks with non-monotonic transfer functions yield an enhanced memory capacity than the conventional monotonic relation. This property of non-monotonic neural networks was also pointed out by Morita *et al* [13] by Monte-Carlo simulation before Nishimori and Opris [12]. The reason why the optimal storage capacity of a non-monotonic transfer function increases is that a weak value of the total input to a neuron implies a confused state and an inverted output for a weak input might work as a trial toward an improved retrieval.

In this paper we investigate the retrieval phase diagram of Hopfield networks which update asynchronously and have a non-monotonic transfer function by a mean-field theory of statistical mechanics proposed by Geszti [14, 15]. In section 2 we show the formulation of the mean-field approximation to the asynchronous Hopfield networks with non-monotonic transfer function and equations of state are derived. In section 3 we extend our formulation to networks with a general type of transfer function. In section 4 we study the performance of non-monotonic Hopfield networks when their synapses depend on the state of networks using the method proposed by Zertuche *et al* [11]. In section 5 we compare the results of our calculations with the results obtained by Nishimori and Opris [12].

2. Equations of state

Most of the investigations which discussed equilibrium properties of fully connected Hopfield networks by statistical mechanics were restricted to networks with equilibrium free energy. In order to show the existence of such free energy, we must make sure that the synaptic couplings are symmetric and the transfer function is monotonic. Our model in this paper has symmetric couplings, but the transfer function is non-monotonic. In order to overcome this difficulty we use the mean-field approximation as follows.

Let us suppose that the ith neuron updates its state according to the probability [7]

$$\text{Prob}(\sigma_i(t+1)) = \tfrac{1}{2}[1 + \sigma_i(t+1)f(h_i^t)] \tag{1}$$

where $\sigma_i = \pm 1$, $\xi_i^\mu = \pm 1$, and the local field to the ith neuron h_i^t is defined as

$$h_i^t = \frac{1}{N} \sum_{j \neq i} \sum_\mu \xi_i^\mu \xi_j^\mu \sigma_j(t). \tag{2}$$
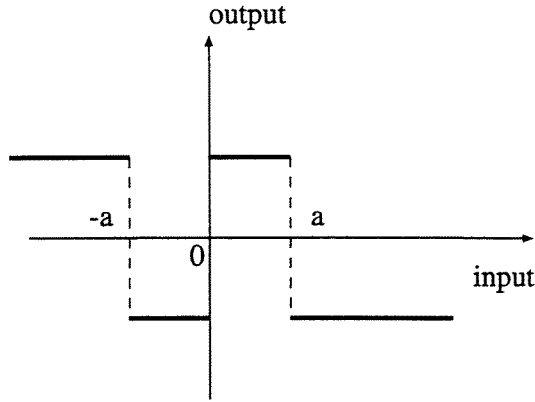
Figure 1. The stepwise-type non-monotonic transfer function.

Then we can calculate the average value of $\sigma_i(t+1)$ as

$$\langle \sigma_i(t+1) \rangle = (+1) \times \tfrac{1}{2}[1 + f(h_i^t)] + (-1) \times \tfrac{1}{2}[1 - f(h_i^t)] = f(h_i^t). \qquad (3)$$

For the equilibrium state we obtain the equation of state by mean-field approximation [14] as

$$\langle \sigma_i \rangle = f\left(\frac{1}{N} \sum_j \sum_\mu \xi_i^\mu \xi_j^\mu \langle \sigma_j \rangle \right). \qquad (4)$$

In this section we choose the function $f(x)$ as

$$f(x) = \tanh(-\beta(x+a)) + \tanh(-\beta(x-a)) + \tanh(\beta x) \qquad (5)$$

where a is a positive constant and β is a parameter related to the synaptic noise. This non-monotonic transfer function reduces to the form in figure 1 in the limit $\beta = 1/T \to \infty$. Then equation (4) can be rewritten explicitly as

$$\langle \sigma_i \rangle = \tanh\left[-\frac{\beta}{N} \sum_j \sum_\mu \xi_i^\mu \xi_j^\mu \langle \sigma_j \rangle - \beta a \right] + \tanh\left[-\frac{\beta}{N} \sum_j \sum_\mu \xi_i^\mu \xi_j^\mu \langle \sigma_j \rangle + \beta a \right]$$

$$+ \tanh\left[\frac{\beta}{N} \sum_j \sum_\mu \xi_i^\mu \xi_j^\mu \langle \sigma_j \rangle \right]. \qquad (6)$$

We introduce the overlap between the equilibrium state of the network $\langle \sigma_i \rangle$ and an embedded pattern ν as follows:

$$m_\nu \equiv \frac{1}{N} \sum_i \xi_i^\nu \langle \sigma_i \rangle. \qquad (7)$$

Using this overlap parameter, we may rewrite (6) as

$$m_\nu = \frac{1}{N} \sum_i \xi_i^\nu \left\{ \tanh\left[\beta\left(-\sum_\mu \xi_i^\mu m_\mu + a \right) \right] + \tanh\left[\beta\left(-\sum_\mu \xi_i^\mu m_\mu - a \right) \right] \right.$$

$$\left. + \tanh\left[\beta \sum_\mu \xi_i^\mu \right] \right\}. \qquad (8)$$

We divide the term $\sum_\mu \xi_i^\mu m_\mu$ appearing above into three parts: the first for $\mu = 1 \,(\neq \nu)$ which corresponds to the retrieved state, the second corresponding to the term $\mu = \nu$, and

the rest. Then we get

$$m_\nu = -\frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh\left[\beta(m_1 + \xi_i^\nu \xi_i^1 m_\nu + \sum_{\mu \neq 1, \nu} \xi_i^\mu \xi_i^1 m_\mu - a)\right]$$

$$-\frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh\left[\beta(m_1 + \xi_i^\nu \xi_i^1 m_\nu + \sum_{\mu \neq 1, \nu} \xi_i^\mu \xi_i^1 m_\mu + a)\right]$$

$$+\frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh\left[\beta(m_1 + \xi_i^\nu \xi_i^1 m_\nu + \sum_{\mu \neq 1, \nu} \xi_i^\mu \xi_i^1 m_\mu)\right]. \tag{9}$$

Here, m_1 is of order 1, and the summation $\sum_{\mu \neq 1, \nu} \xi_i^\mu \xi_i^1 m_\mu$ is also of order 1, while the term $\xi_i^\nu \xi_i^1 m_\nu$ is much smaller, $\mathcal{O}(1/\sqrt{N})$. Then we may expand the function tanh appearing in (9) to first order of $\xi_i^\nu \xi_i^1 m_\nu$. The terms $\sum_{\mu \neq 1, \nu} \xi_i^\mu \xi_i^1 m_\mu$ appearing in tanh may be regarded as Gaussian variables with mean zero and variance $\sum_{\mu \neq 1, \nu} m_\mu^2 \equiv \alpha r$ [14]. Under this approximation, we may replace the summation $(1/N) \sum_i$ by a Gaussian integral. We next square the equation (9) to calculate $r = \sum_{\nu \neq 1} m_\nu^2/\alpha$. Following the procedure introduced by Gesti [14], we obtain the equations of state in the limit of $N \to \infty$ as follows:

$$r = [1 - \beta(q_+ + q_- - q - 1)]^{-2} \tag{10}$$

and

$$m = -\int Dz \, \tanh[\beta(m + \sqrt{\alpha r} z - a)] - \int Dz \, \tanh[\beta(m + \sqrt{\alpha r} z + a)]$$

$$+ \int Dz \, \tanh[\beta(m + \sqrt{\alpha r} z)] \tag{11}$$

where we set $m_1 = m$ and introduced the Edwards–Anderson-like [16] order parameters

$$q \equiv \int Dz \, \tanh^2[\beta(m_1 + \sqrt{\alpha r} z)] \tag{12}$$

$$q_\pm \equiv \int Dz \, \tanh^2[\beta(m + \sqrt{\alpha r} z \pm a)]. \tag{13}$$

In the zero-temperature limit, these order parameters can be rewritten as

$$q = 1 - \sqrt{\frac{2}{\pi \alpha r \beta^2}} \exp\left(-\frac{m^2}{2\alpha r}\right) \tag{14}$$

$$q_\pm = 1 - \sqrt{\frac{2}{\pi \alpha r \beta^2}} \exp\left(-\frac{(m \pm a)^2}{2\alpha r}\right). \tag{15}$$

Then equations of state lead to

$$r = \left[1 + \sqrt{\frac{2}{\pi \alpha r}} \left\{\exp\left(-\frac{(m + a)^2}{2\alpha r}\right) + \exp\left(-\frac{(m - a)^2}{2\alpha r}\right) + \exp\left(-\frac{m^2}{2\alpha r}\right)\right\}\right]^{-2} \tag{16}$$

and

$$m = -\operatorname{erf}\left(\frac{m - a}{\sqrt{2\alpha r}}\right) - \operatorname{erf}\left(\frac{m + a}{\sqrt{2\alpha r}}\right) + \operatorname{erf}\left(\frac{m}{\sqrt{2\alpha r}}\right) \tag{17}$$

where

$$\operatorname{erfc}(x) \equiv 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt \, \exp(-t^2). \tag{18}$$
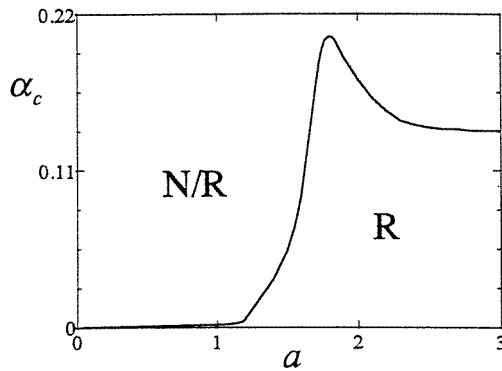
Figure 2. Phase diagram for the non-monotonic transfer function in figure 1 obtained by the mean-field approximation. Retrieval is successful in the region R (normal retrieval) and unsuccessful in N/R. $\alpha_c(a = \infty) = 0.138$ (consistent with Amit *et al*) and $\alpha_c(a = 1.77) = 0.211$ (maximum value).

For $a \to \infty$ one recovers the equations of state of the Hopfield model at $T = 0$ obtained by Amit *et al* [3, 4]].

The neural network is useful as an associative memory as long as the mean-field equations (10) and (11) have a solution of the form $m_\mu = (m, 0, \ldots, 0, 0)$ with $m \neq 0$. At $T = 0$, there exist metastable states which are highly correlated with particular embedded patterns as long as $p = \alpha N < \alpha_c N$. We solved equations (16) and (17) numerically and obtained the critical capacity α_c. This result is plotted as a function of the parameter a in figure 2. In this phase diagram, the region R denotes the retrieval phase and the region N/R means the non-retrieval phase where the self-consistent equations (16) and (17) do not have a non-zero solution of m. From this result, we see that Hopfield networks with non-monotonic transfer function store more patterns than the networks with monotonic transfer function [3]: α_c has the maximum value 0.211 at $a = 1.77$. The shape of the $\alpha_c(a)$ curve has similar properties, in the following sense, with that of Nishimori and Opris [12] who calculated this critical curve by the equilibrium relation of Amari–Maginu dynamics for synchronous networks:

- There exists a certain value of a that maximizes α_c.
- α_c approaches the monotonic value (0.138 in the present case) in the limit $a \to \infty$.

An interesting observation is that an iterative solution of the self-consistent equations (16) and (17) showed oscillatory behaviour in a restricted region around $a = 0$ and $\alpha = 0$ in the phase diagram. In consideration of similar observations from dynamical treatments [12], such a phase should be characterized by time development of the overlap m_t in the sense that $m_{t+1} > 0$ if $m_t < 0$ and $m_{t+1} < 0$ if $m_t > 0$. However, we should be warned that, strictly speaking, as we treat the static properties of the non-monotonic Hopfield model, we cannot extract the dynamical behaviour of m_t from our formulation.

Next, in order to get the α–T phase diagram, we solved the finite-temperature self-consistent equations (10) and (11). For the case of $a = 1.80$ we plotted the α–T curve in figure 3. We have found that the transition from the (normal) retrieval phase to the spin-glass phase is of first order.
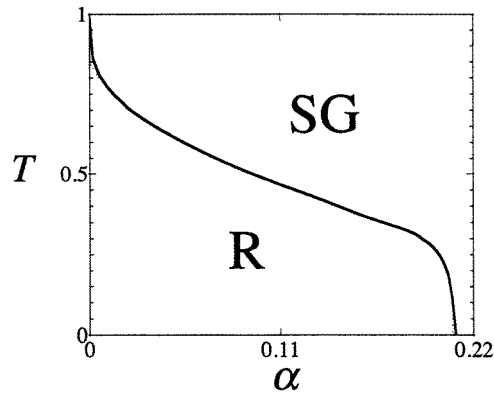
Figure 3. The α–T phase diagram of the Hopfield model with non-monotonic transfer function ($a = 1.80$). Retrieval is successful in R (retrieval phase) and unsuccessful in SG (spin-glass phase).

3. Extension to a general transfer function

In this section we show that our formulation can be extended to the networks with a general transfer function [12]. The mean-field equation of state for a general transfer function f is already given in (4). Equation (9) of the parameter m_ν can be rewritten for this general transfer function as follows.

$$m_\nu = \frac{1}{N} \sum_i \xi_i^\nu f\left(\sum_\mu \xi_i^\mu m_\mu\right). \tag{19}$$

Here we have included the effect of the control parameter β, corresponding to the thermal noise, in the general function f. We should not forget that the absolute value of the function f does not exceed 1 because otherwise probabilistic interpretation (1) does not make sense. For the general transfer function f, we can obtain the equations of state in the same way as in the previous section. The result is

$$r = \frac{Q_0}{[1 - Q]^2} \tag{20}$$

and

$$m = \int Dz\, f(m + \sqrt{\alpha r}z) \tag{21}$$

where

$$Q_0 \equiv \int Dz\, f^2(m + \sqrt{\alpha r}z) \tag{22}$$

$$Q \equiv \int Dz\, f'(m + \sqrt{\alpha r}z). \tag{23}$$

Note that by setting $f(x) = \tanh(\beta x)$ we recover the result by Amit *et al* [3].

We next show that our equations of state for the general transfer function are different from the result of Nishimori and Opris [12]. They calculated the recursion relations of macro-variables m_t and σ_t (the latter being the measure of disturbance from non-retrieved patterns) by generalizing the Amari–Maginu-type signal-to-noise ratio analysis

[6] to stochastic dynamical process in the case of synchronous dynamics. Their result is

$$m_{t+1} = \int \mathrm{D}z \ f(m_t + \sigma_t z) \tag{24}$$

$$\sigma_{t+1}^2 = \alpha + 2\alpha m_t m_{t+1} h(m_t, \sigma_t) + \sigma_t^2 h^2(m_t, \sigma_t) \tag{25}$$

where h is defined by

$$h(m_t, \sigma_t) = \int \mathrm{D}z \ f'(m_t + \sigma_t z). \tag{26}$$

Taking the equilibrium limit $t \to \infty$ and setting $\sigma_\infty = \sqrt{\alpha r}$ and $m_\infty = m$, we get equations of state with respect to m and r as

$$m = \int \mathrm{D}z \ f(m + \sqrt{\alpha r}z) \tag{27}$$

$$r = \frac{1 + \alpha m^2 h(m, \sqrt{\alpha r})}{[1 - \{h(m, \sqrt{\alpha r})\}^2]} \tag{28}$$

where

$$h(m, \sqrt{\alpha r}) = \int \mathrm{D}z \ f'(m + \sqrt{\alpha r}z). \tag{29}$$

This is different from our result (20) and (21). We may suppose that this difference comes from the difference between synchronous and asynchronous dynamics. There is no *a priori* reason why the equilibrium properties of synchronous networks should coincide with those with asynchronous networks.

4. State-dependent synapses

As long as the number of embedded patterns satisfies $p \ll N$, the noise term $\sum_{\mu \neq 1, v} \xi_i^\mu \xi_i^\mu m_\mu$ appearing in the mean-field equation is of order $\mathcal{O}(1/N)$ and we can neglect this term. However, if $p = \alpha N$ with α finite, this same term becomes $\mathcal{O}(1)$ and this contribution cannot be neglected. These non-retrieved memories ξ^μ ($\mu \neq 1, v$), which appear in $\sum_{\mu \neq 1, v} \xi_i^\mu \xi_i^\mu m_\mu$, prevent networks from retrieving the embedded pattern. The storage capacity of the Hopfield model is limited by the contribution of a large number of weakly correlated patterns. For the conventional Hopfield model, patterns $\{\xi^\mu\}$ are stored by the Hebbian-type synaptic interaction $J_{ij} = (1/N) \sum_\mu \xi_i^\mu \xi_j^\mu$. Therefore, in order to exclude non-retrieved memories which have small overlaps with the state of the network, we should modify the synaptic interaction so that only patterns with large overlaps with state contribute to the Hebbian rule [11]. Our main interest in this section is to what degree the stability of the memorized states is improved by this state-dependent synaptic interaction [11, 17] and how many patterns are stored in the Hopfield networks with non-monotonic transfer function.

We use the next state-dependent synapses by introducing a threshold η ($\geqslant 0$) [11]:

$$J_{ij} = \frac{1}{N} \sum_\mu \xi_i^\mu \xi_j^\mu \Theta \left(m_\mu^2 - \frac{\eta^2}{N} \right). \tag{30}$$

The factor of the step function $\Theta(x)$ means that an embedded pattern ξ^μ is excluded if the overlap between the pattern and network state m_μ is below a threshold $m_\mu^2 < \eta^2/N$. We expect that the performance of a network as an associative memory is improved by introducing this type of synapses with threshold $\eta > 0$ to exclude the spurious memories

disturbing retrieval. We introduced the factor $1/N$ because m_μ is of order $1/\sqrt{N}$. It is important to bear in mind that the conventional Hebb interaction $J_{ij} = (1/N) \sum_\mu \xi_i^\mu \xi_i^\mu$ is recovered by setting $\eta = 0$. Using this coupling, we rewrite the mean-field equation obtained in section 2 as follows:

$$m_\nu = \frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh[\beta(m + \eta_i^\mu)] + \beta m_\nu \Theta\left(m_\nu^2 - \frac{\eta^2}{N}\right) \frac{1}{N} \sum_i [1 - \tanh^2 \beta(m + \eta_i^\mu)]$$

$$-\frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh[\beta(m + \eta_i^\mu - a)]$$

$$-\beta m_\nu \Theta\left(m_\nu^2 - \frac{\eta^2}{N}\right) \frac{1}{N} \sum_i [1 - \tanh^2 \beta(m + \eta_i^\mu - a)]$$

$$-\frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh[\beta(m + \eta_i^\mu + a)]$$

$$-\beta m_\nu \Theta\left(m_\nu^2 - \frac{\eta^2}{N}\right) \frac{1}{N} \sum_i [1 - \tanh^2 \beta(m + \eta_i^\mu + a)] \tag{31}$$

where we introduced ζ_i^μ as follows:

$$\zeta_i^\mu \equiv \sum_{\mu \neq 1, \nu} \xi_i^\mu \xi_i^1 m_\mu \Theta\left(m_\mu^2 - \frac{\eta^2}{N}\right). \tag{32}$$

This is the sum of a large number ($= \alpha N$) of small terms (of order $1/N$). We now assume that the small contributions m_μ ($\mu \neq 1$) have identical Gaussian distributions centred at zero, with variance σ^2/N. Strictly speaking, this statement is not exact because m_μ are related through (31). Nevertheless we accept this approximation in this paper. By the same arguments, ζ_j^μ is assumed to have a Gaussian distribution with variance αr and average zero, so that

$$\alpha r = \langle\!\langle (\eta_i^\nu)^2 \rangle\!\rangle. \tag{33}$$

We also introduce the Edwards–Anderson like order parameters [16] as

$$q \equiv \frac{1}{N} \sum_i \tanh^2 \beta(m + \eta_i^\mu) = \int Dz \, \tanh^2 \beta(m + \sqrt{\alpha r} z) \tag{34}$$

$$q_\pm \equiv \frac{1}{N} \sum_i \tanh^2 \beta(m + \eta_i^\mu \pm a) = \int Dz \, \tanh^2 \beta(m + \sqrt{\alpha r} z \pm a). \tag{35}$$

Using these parameters, (31) leads to

$$m_\nu \left[1 - \beta(q_+ + q_- - q - 1)\Theta\left(m_\nu^2 - \frac{\eta^2}{N}\right)\right] = \frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh[\beta(m + \eta_i^\mu)]$$

$$-\frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh[\beta(m + \eta_i^\mu - a)]$$

$$-\frac{1}{N} \sum_i \xi_i^\nu \xi_i^1 \tanh[\beta(m + \eta_i^\mu + a)]. \tag{36}$$

Squaring this expression and averaging it over the distribution of patterns, we get

$$\sigma^2 + \{[1 - \beta(q_+ + q_- - q - 1)]^2\} r = 1. \tag{37}$$

Equations (32) and (33) lead to

$$\alpha r = p \left\langle\!\!\left\langle m_\mu^2 \Theta\left(m_\mu^2 - \frac{\eta^2}{N}\right)\right\rangle\!\!\right\rangle = p \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi\sigma^2/N}} \exp\left(-\frac{Nz^2}{2\sigma^2}\right) z^2 \Theta\left(z^2 - \frac{\eta^2}{N}\right). \tag{38}$$

Using the transformation $Nz^2/2\sigma^2 = t$, we find

$$\alpha r = \frac{p}{N}\left(\frac{2}{\sqrt{\pi}}\right)\sigma^2 \int_{\eta^2/2\sigma^2}^{\infty} t^{-\frac{1}{2}} \exp(-t)\, dt = \alpha \frac{2}{\sqrt{\pi}} \sigma^2 \Gamma\left(\frac{3}{2}, \frac{\eta^2}{2\sigma^2}\right). \tag{39}$$

The final expression of r is

$$r = \frac{2}{\sqrt{\pi}}\sigma^2 \Gamma\left(\frac{3}{2}, \frac{\eta^2}{2\sigma^2}\right) \tag{40}$$

where Γ is the incomplete gamma function defined as

$$\Gamma(z, p) = \int_p^{\infty} \exp(-t)t^{z-1}\, dt. \tag{41}$$

Another equation is obtained for m by taking $\mu = 1$ in (31) in the limit $\beta \to \infty$. This result agrees with (11) in section 2. The term $\beta(q_+ + q_- - q - 1)$ appearing in (37) leads in the limit $\beta \to \infty$ to

$$C \equiv \beta(q_+ + q_- - q - 1)$$
$$= -\sqrt{\frac{2}{\pi\alpha r}}\left\{\exp\left(-\frac{(m+a)^2}{2\alpha r}\right) + \exp\left(-\frac{(m-a)^2}{2\alpha r}\right) - \exp\left(-\frac{m^2}{2\alpha r}\right)\right\}. \tag{42}$$

Finally we have the equations of state as follows:

$$\sigma^2 + [(1-C)^2 - 1]r = 1 \tag{43}$$

$$r = \frac{2}{\sqrt{\pi}}\sigma^2 \Gamma\left(\frac{3}{2}, \frac{\eta^2}{2\sigma^2}\right) \tag{44}$$

$$m = -\operatorname{erf}\left(\frac{m-a}{\sqrt{2\alpha r}}\right) - \operatorname{erf}\left(\frac{m+a}{\sqrt{2\alpha r}}\right) + \operatorname{erf}\left(\frac{m}{\sqrt{2\alpha r}}\right). \tag{45}$$

For simplicity we use a variable k defined by

$$k = \frac{\sqrt{\pi}}{2} \frac{1}{\Gamma(\frac{3}{2}, \frac{\eta^2}{2\sigma^2})} - 1. \tag{46}$$

Equations (43), (44) and (46) are written as

$$r = \frac{1}{(1-C)^2 + k} \tag{47}$$

$$\sigma^2 = \frac{1+k}{(1-C)^2 + k}. \tag{48}$$

For $\eta = 0$, one has $k = 0$ and from (43), (44) and (46) one recovers the equations for the non-monotonic Hopfield model at $T = 0$ discussed in the previous section.

We evaluated equations (45), (46) and (43) and obtained the optimal storage capacity α_c. We show the results for α_c as a function of η for the case $T = 0$, $a = 3.0$ in figure 4. A similar result is plotted in figure 5 for the case of $T = 0$ and $a = 1.80$. We also show the parameter-a dependence of the capacity α_c for the cases of $\eta = 1.0$, 0.8 and 0.6 in figure 6. From this figure we see that as the threshold parameter η increases, more patterns can be embedded by the modified Hebbian rule (30). It is observed how the storage
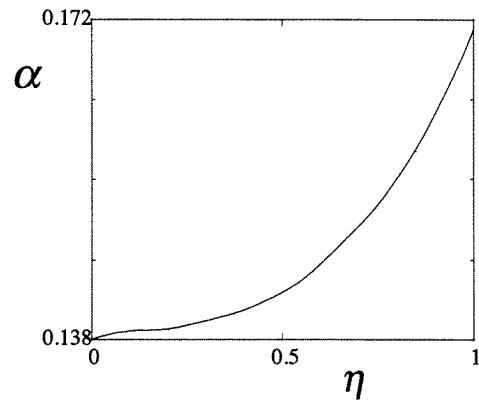
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Figure 4. A slice of the phase space for the non-monotonic networks with threshold η at $T = 0$ and $a = 3.0$. The Hopfield value $\alpha_c = 0.138$ is found at $\eta = 0$; for a threshold η equal to 1 the optimal capacity increases to $\alpha_c = 0.171$.
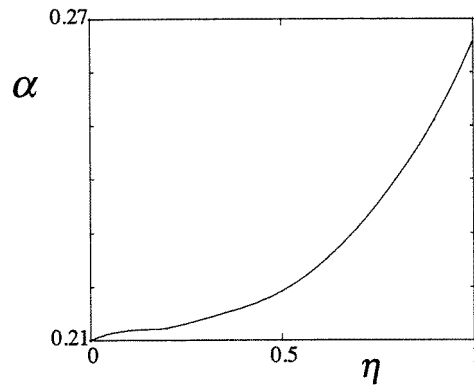
Figure 5. A slice of the phase space for the non-monotonic networks with threshold η at $T = 0$ and $a = 1.8$.
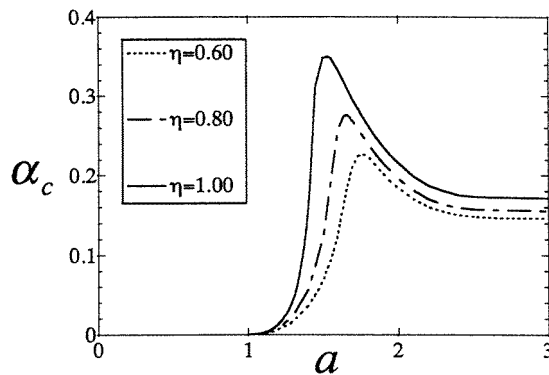

Figure 6. Optimal storage capacity $\alpha_c(a)$ of the non-monotonic network with thresholds $\eta = 1.0$, 0.8 and 0.6 at $T = 0$.

capacity of the non-monotonic Hopfield networks is improved as the value of η increases from $\eta = 0$. Therefore using the neural networks with non-monotonic transfer function and state-dependent synapses, we can get an associative memory with the high-quality performance. We also show the overlap m as a function of α for the case of $\eta = 0.80$, $a = 3.00$ and $T = 0$, $\eta = 0.80$, $a = 1.60$ and $T = 0$ in figures 7 and 8, respectively. From these figures $m(\alpha)$ is seen to drop to zero discontinuously at the critical capacity $\alpha_c \sim 0.155$ for $a = 3.0$ and $\alpha_c \sim 0.257$ for $a = 1.6$.

5. Discussion

We have investigated the retrieval phase diagrams by the mean-field approximation [14] in the Hopfield networks with asynchronous dynamics. Mean-field approximation was extended to the general type of transfer function. The result shows that a non-monotonic transfer function yields an enhanced memory capacity for a around 0.211. This confirms the claim of Morita *et al* [13] who used the numerical simulation for synchronous dynamics
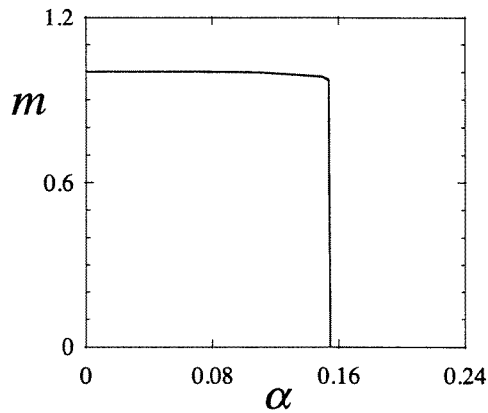
Figure 7. The order parameter $m(\alpha)$ at $T = 0$ and $\eta = 0.80$ and $a = 3.00$.
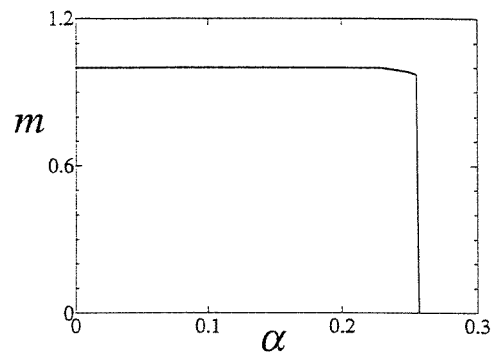
Figure 8. The order parameter $m(\alpha)$ at $T = 0$ and $\eta = 0.80$ and $a = 1.60$.

and the result of Nishimori and Opris [12] who used the equilibrium relation of the Amari and Maginu [6] dynamics for synchronous dynamics. The properties of the phase diagram obtained in this paper qualitatively resemble those of the phase diagram of the synchronous neural networks. It is interesting that our calculation for the asynchronous network also showed the enhancement of the capacity as in the synchronous case: the shape of the retrieval phase diagram in this paper is similar to that of Nishimori and Opris [12]. A difference is that within our formulation of the asynchronous networks, the oscillatory phase (limit-cycle phase) found by Nishimori and Opris was not obtained clearly. This phase is characterized by the behaviour of the dynamical order parameter m_t which is $m_{t+1} > 0$ if $m_t < 0$ and $m_{t+1} < 0$ if $m_t > 0$ in the range of $0 < a < 1$. As we used the equilibrium statistical mechanics to get the phase diagram of the non-monotonic Hopfield model, we cannot draw definite conclusions about the dynamical order parameter m_t.

For the non-monotonic Hopfield model, the property of asynchronous dynamics is an open problem. However, the oscillatory behaviour during the process of recursion-type solution of equilibrium equations of state may be related to the dynamical oscillatory phase found in the same region of the phase diagram.

We extended our formulation to the general transfer function in this paper. It is interesting to investigate whether the storage capacity is enhanced by a transfer function which has a different shape from the stepwise-type one. For the moment, we could not find better transfer functions than the stepwise-type one. And we also showed that the Hopfield network with non-monotonic transfer function and state-dependent couplings can store a large number of patterns. From these results one can confirm that the limit of an associative memory with the Hebb-type interactions consists in the effects of spurious states each of which has a small correlation with embedded patterns.

We may be able to find a new type of network which shows better performance than the conventional one by introducing non-monotonic transfer function and state-dependent synapses.

Acknowledgments

of Hopfield networks and the Amari–Maginu theory.

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